PRESSURE PULSE ON THE BOUNDARY OF AN ELASTIC INHOMOGENEOUS HALF-PLANE

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The problem about the motion of a pressure pulse at constant velocity along the boundary of an elastic homogeneous half-plane has been examined in [1-3]. The problem was considered as stationary in [1, 2], while in [3] it was solved by using a Laplace time transformation. An analogous problem is considered in this paper for an elastic half-plane with variable Lamé parameters and density of the medium.

1. An elastic half-plane xz, z > 0 is considered whose Lamé parameters λ, μ and density ρ depend on the coordinate z according to the power law

$$\lambda = \lambda_0\varepsilon^\nu, \quad \mu = \mu_0\varepsilon^\nu, \quad \rho = \rho_0\varepsilon^\omega$$
$$\varepsilon = az + 1, \quad \nu = \frac{\gamma+1}{\gamma-1}, \quad \omega = \frac{4}{\gamma-1}, \quad \gamma = \frac{\lambda+2\mu}{\mu} \tag{1.1}$$

As is shown in [4], the equation of motion of such a medium can be represented as

$$\left[\nabla^2 + \frac{4a\partial}{(\gamma-1)\,\varepsilon\partial z} - v_n^{-2}\varepsilon^{(3-\gamma)/(\gamma-1)}\frac{\partial^2}{\partial t^2}\right]\psi_n = 0 \tag{1.2}$$
$$v_1^2 = (\lambda_0 + 2\mu_0)\rho_0^{-1}, \quad v_2^2 = \mu_0\rho_0^{-1}, \quad n = 1, 2$$

where v_1, v_2 are the elastic wave velocities, and a is a dimensionless parameter.

The functions ψ_n and displacements u_n are related by the dependences

$$f_1\mathbf{u}_1 = \nabla(f_1\psi_1), \quad f_2\mathbf{u}_2 = \nabla\times\left(i_y f_2\frac{\partial\psi_2}{\partial x}\right) \tag{1.3}$$

The weight functions f_n depends only on z; the unit vector i_y is directed along the y axis. Let us consider a medium for which $\lambda_0 = \mu_0$. Then γ = 3, and (1.2) simplify and become

$$\left(\nabla^2 + \frac{2a\partial}{\varepsilon\,\partial z} - v_n^{-2}\frac{\partial^2}{\partial t^2}\right)\psi_n = 0 \tag{1.4}$$

For zero initial data the solution of the system (1.4) should satisfy the boundary conditions

$$\sigma_z = -\delta(vt - x), \quad \tau_{xz} = 0 \quad \text{for} \quad z = 0 \tag{1.5}$$

Here δ(α) is the Dirac function, v is the velocity of pressure pulse motion, t is the time, and σ_z, τ_{xz} are stress tensor components.

Let us introduce a new variable s by means of the formula s = vt − x. Then the solution of the system (1.4) can be represented in the form

$$\psi_1 = \varepsilon^{-1}\int_{-\infty}^{\infty} G(\alpha)\exp(i\alpha s - \alpha\eta z)\,d\alpha$$
$$\psi_2 = \varepsilon^{-1}\int_{-\infty}^{\infty} Q(\alpha)\exp(i\alpha s - \alpha\delta z)\,d\alpha$$
$$\eta = (1 - v^2v_1^{-2}\alpha^{-2})^{1/2}, \quad \delta = (1 - v^2v_2^{-2}\alpha^{-2})^{1/2}$$

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where α is the separation parameter of (1.4).

Let us use the boundary conditions (1.5) and the dependence (1.3) to determine the functions G and Q. This reduces to two equations,

$$\begin{aligned} &G(2\eta + a) + Q(2\alpha^2 - v^2 v_2^{-2} + a\delta) = 0 \\ &G(2\alpha^2 - v^2 v_2^{-2} + 3a\eta) + Q(2\delta + 3a)\alpha^2 = -2\pi^{-1} \end{aligned} \tag{1.6}$$

An integral representation of the Dirac function

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha$$

was used to obtain the second equation.

Having determined the constants from (1.6), we find the displacements by using (1.3)

$$u_x = \frac{\mathrm{Im}}{2\pi\varepsilon} \int_0^{\infty} \alpha\,[(\zeta + a\delta)\, e^{-\eta z} - \delta\,(2\eta + a)\, e^{-\delta z}]\, R^{-1} e^{-i\alpha s}\, d\alpha$$

$$u_z = \frac{\mathrm{Re}}{2\pi\varepsilon} \int_0^{\infty} [\alpha^2 (2\eta + a)\, e^{-\delta z} - (\zeta + a\delta)\, \eta e^{-\eta z}]\, R^{-1} e^{-i\alpha s}\, d\alpha$$

The Rayleigh function of an inhomogeneous medium is denoted by R, of a homogeneous medium by R_0, and $\zeta = \varkappa + iw$ is the complex variable of integration

$$R = 4\eta\delta\alpha^2 - \zeta^2 - av^2 v_2^{-2}\alpha^2(3\eta + \delta) + 3a^2(\alpha^2 - \eta\delta)$$

$$\zeta = 2\alpha^2 - v^2 v_2^{-2}$$

2. Let us assume that the conditions

$$\sigma_z = -H(vt - x), \quad \tau_{xz} = 0 \quad \text{for} \quad z = 0 \tag{2.1}$$

are given on the half-plane boundary.

Here $H(\alpha)$ is the Heaviside function.

Applying a Laplace time transform to the equations of motion (1.4) and the boundary conditions (2.1) and using (1.3), we obtain

$$\left(\nabla^2 + \frac{2a\partial}{\varepsilon\partial z} - v_n^{-2} p^2\right)\bar{\psi}_n = 0, \quad \bar{\psi}(p) = \int_0^{\infty} e^{-pt}\psi(t)\, dt \tag{2.2}$$

$$\begin{aligned} &\bar{\sigma}_z = \frac{\mu}{\varepsilon}\left[\left(\frac{\partial^2}{\partial x^2} + 3\frac{\partial^2}{\partial z^2} - \frac{3a\partial}{\varepsilon\partial z}\right)\bar{\psi}_1 + \frac{\partial^2}{\partial x^2}\left(2\frac{\partial}{\partial z} - \frac{3a}{\varepsilon}\right)\bar{\psi}_2\right]_{z=0} = -\frac{e^{-pxH(x)}}{p} \\ &\bar{\tau}_{xz} = \frac{\mu\partial}{\varepsilon\,\partial x}\left[\left(2\frac{\partial}{\partial z} - \frac{a}{\varepsilon}\right)\bar{\psi}_1 + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} + \frac{a\partial}{\varepsilon\,\partial z}\right)\bar{\psi}_2\right]_{z=0} = 0 \end{aligned} \tag{2.3}$$

where p is the complex Laplace transform parameter.

Let us seek the solution of (2.2) in the form

$$\begin{aligned} &\bar{\psi}_1 = \varepsilon^{-1} \int_{-\infty}^{\infty} G(\zeta) \exp(ip\zeta x - p\sqrt{\zeta^2 + G^2}\, z)\, d\zeta, \quad c_1 = v^{-1} \\ &\bar{\psi}_2 = \varepsilon^{-1} \int_{-\infty}^{\infty} Q(\zeta) \exp(ip\zeta x - p\sqrt{\zeta^2 + c_2^2}\, z)\, d\zeta, \quad c_2 = v_2^{-1}, \quad \zeta = \alpha p^{-1} \end{aligned} \tag{2.4}$$

The integration is over the real axis of the complex ζ plane. For uniqueness of the intergrands, let us fix the branches of the roots by the condition $\sqrt{1} = 1$. Substituting (2.4) into (2.3), we arrive at two integral equations to determine $G(\zeta)$, $Q(\zeta)$,

$$\begin{aligned} &\int_{-\infty}^{\infty} [G(2p\delta + a) + Q(p^2\eta^2 + p^2\zeta^2 - ap\eta)] \exp(ip\zeta x)\, d\zeta = 0 \\ &\int_{-\infty}^{\infty} [G(-p\zeta^2 + 3p\delta^2 + 3a\delta) + Q(2\eta p^2\zeta^2 + 3ap\zeta^2)]\, e^{ip\zeta x} = e^{-pxH(x)}/p^2 \end{aligned} \tag{2.5}$$

Setting

$$G(\zeta) = -F(\zeta)(p^2\eta^2 + p^2\zeta^2 + ap\eta)$$

$$Q(\zeta) = F(\zeta)(2p\delta + a)$$

$$\delta = \sqrt{\zeta^2 + c_1^2}, \quad \eta = \sqrt{\zeta^2 + c_2^2}$$

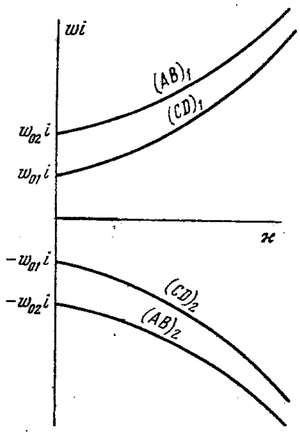

Fig. 1

We satisfy the first equation in (2.5) identically and obtain an equation to determine the function F(ζ)

$$\int_{-\infty}^{\infty} F(\zeta) R(\zeta) \exp(ip\zeta x)\, d\zeta = -p^{-3} \exp\left(-\frac{pxH(x)}{v}\right) \tag{2.6}$$

$$R(\zeta) = p^2 [4\zeta^2\eta\delta - (2\zeta^2 + 1)^2] - ap(3\delta + \eta) + 3a^2 \times \\ \times (1 - \eta\delta) = [4\zeta^2\eta\delta - (2\zeta^2 + 1)^2](p + p_1)(p + p_2) \tag{2.7}$$

Closing the contour of integration in (2.6) for x > 0 and x < 0, in the upper and lower half-planes, respectively, we find that the equation is satisfied for x > 0 if

$$F(\zeta) R(\zeta) = L_+(\zeta) [2\pi i p^3 L_+(iv^{-1})(\zeta - iv^{-1})]^{-1} \tag{2.8}$$

and for x < 0

$$F(\zeta) R(\zeta) = -L_-(\zeta) [2\pi i p^3 L_-(-i\beta)(\zeta + i\beta)]^{-1}, \quad \beta \to \infty$$

On the real axis we have

$$\lambda_+ = \frac{L_+(\xi + i\beta)}{L_+(iv^{-1})} = -\frac{L_-(\xi)(\xi - iv^{-1})}{L_-(-i\beta)} = \lambda_-$$

Here L_+, L_- are analytic functions, without singularities in the upper and lower half-planes, respectively.

It hence follows that λ_+ is an analytic continuation of λ_- in the upper half-plane. Hence, λ is an analytic function in the whole plane, i.e., is entire.

For the integrals (2.5) to converge it is necessary to require that for z = 0

$$\lambda_-(\zeta) = A\zeta + B \tag{2.9}$$

The constants A and B are determined by means of the values $\lambda_-(-i\beta)$, $\lambda_-(iv^{-1})$. Having the expression for $\lambda_-(\zeta)$, we find F(ζ) from (2.9). Then (2.6) will yield the desired functions G(ζ), Q(ζ). By using (2.4) and (1.3) we determine the displacements as the transforms

$$\bar{u}_x = \frac{\mathrm{Im}}{2\pi\varepsilon} \int_{-\infty}^{\infty} \frac{e^{ip\zeta x}}{p(\zeta - iv^{-1}) R_0} \left\{ \left[2\delta\eta + \frac{pa[\eta R_0 + 2\delta\eta(3\delta + \eta)] - 2a^2 s_1}{R} \right] e^{-p\eta z} - \right.$$
$$\left. - \left[(2\zeta^2 + 1) + \frac{pa[\eta R_0 + (2\zeta^2 + 1)(3\delta + \eta)] - a^2(2\zeta^2 + 1)(1 - \eta\delta)}{R} \right] e^{-p\delta z} \right\} d\zeta$$
$$\bar{u}_z = \frac{\mathrm{Re}}{2\pi\varepsilon} \int_{-\infty}^{\infty} \frac{e^{ip\zeta x}}{p\zeta(\zeta - iv^{-1}) R_0} \left\{ - \left[2\delta\zeta^2 + \frac{pa[\zeta^2 R_0 - 2\delta\zeta^2(3\delta + \eta)] - 2a^2 s_2}{R} \right] e^{-p\eta z} + \right. \tag{2.10}$$
$$\left. + \left[(2\zeta^2 + 1)\delta + \frac{pa[(3\delta + \eta)\delta(2\zeta^2 + 1) + R_0\eta\delta] - a^2(1 - \eta\delta)(2\zeta^2 + 1)\delta}{R} \right] e^{-p\delta z} \right\} d\zeta$$
$$s_1 = \delta\eta(1 - \eta\delta), \quad s_2 = \delta\zeta^2(1 - \eta\delta), \quad R_0 = 4\zeta^2\eta\delta - (2\zeta^2 + 1)^2$$

The components are arranged here in powers of the parameter. For $\alpha = 0$ the expressions (2.1) yield the displacements in the transforms for a homogeneous medium.

3. Inversion of (2.10) into the space of originals is made directly from the table of inverse Laplace transforms. It is expedient to deform the path of integration in the complex plane in such a way that curves would be selected as the paths of integration on which the conditions

$$\mathrm{Im}(i\zeta x - \eta z) = 0$$
$$\mathrm{Im}(i\zeta x - \delta z) = 0 \tag{3.1}$$

would be satisfied.

The relationships (3.1) hold a long segment of the imaginary axis

$$\mathrm{Re}\,\zeta = \varkappa = 0$$
$$0 < \mathrm{Im}\,\zeta < \omega_{0n} = \frac{\chi_n^2 H^2}{\sqrt{1 + H^2}}$$

and on the curves $\omega_n(\chi_n)$ we have

$$\omega_n = \sqrt{H^2\varkappa + \frac{\chi_n^2 H^2}{1+H^2}}, \quad H = \frac{x}{z}, \quad \chi_1 = \frac{1}{\sqrt{3}}, \quad \chi_2 = 1 \tag{3.2}$$

Along the path (3.2) we have

$$\sqrt{x^2 + \chi_n^2} = \frac{\omega}{H} + iH\varkappa, \quad \zeta = (\omega^2 - \omega_0^2)^{1/2}$$
$$ix\zeta - z\sqrt{x^2 + \chi_n^2} = z\omega(H + H^{-1})$$

Let us examine the members in (2.10) which originate because of inhomogeneity of the medium.

Performing the appropriate substitutions, the part of the horizontal displacement caused by inhomogeneity of the medium in the transforms can be written as

$$\bar{u}_{xa} = \frac{\mathrm{Im}}{2\pi\varepsilon p R_0^2} \sum_{n=1}^{2} \int_0^{w_{0n}} \left\{ \frac{e^{-p(wx+\eta z)}}{\zeta - iv^{-1}} \left(\frac{pA_1(iw) + aB_1(iw)}{(p+p_1)(p+p_2)} \right) - \right.$$
$$\left. - \frac{[pA_2(iw) + aB_2(iw)]\, e^{-p(wx+\delta z)}}{(p+p_1)(p+p_1)} \right\} dw + \frac{\mathrm{Im}}{2\pi\varepsilon p R_0^2} \sum_{n=1}^{2} \int_{w_{0n}}^{\infty} \left\{ \frac{e^{-pzw(H+H^{-1})}}{(s - iv^{-1})} \times \right. \tag{3.3}$$
$$\left. \times \left(\frac{pA_1(s) + aB_1(s)}{(p+p_1)(p+p_2)} \right) - \frac{[pA_2(s) + aB_2(s)]\, e^{-pzw(H+H^{-1})}}{(p+p_1)(p+p_2)(s - iv^{-1})} \right\} ds$$

We have here introduced the notation

$$A_1 = \eta R_0 + 2\delta\eta\,(3\delta + \eta), \; B_1 = 2\delta\eta\,(1 - \eta\delta)$$
$$A_2 = \eta R_0 + (2\zeta^2 + 1)(3\delta + \eta), \; B_2 = (2\zeta^2 + 1)(1 - \eta\delta)$$

The first integral is taken over a segment of the imaginary axis, where $\zeta = iw$, and the second, over the curve on which

$$\zeta = H^{-1}\sqrt{w_n^2 - w_{0n}^2} + iw_n \equiv s, \quad ds = \left(\frac{w_n}{H(w_n^2 - w_{0n}^2)^{1/2}} + i \right) dw$$
$$p_{1,2} = a\,\{3\delta + \eta \pm [(3\delta + \eta)^2 - 12\,(\delta\eta - x^2)\,R_0]^{1/2}\}\,(2R_0)^{-1}$$

The paths of integration are indicated in Fig. 1. The two values $\varkappa = 1, {}^1/_3$ correspond to the curves $(AB)_n$ and $(DC)_n$. The expression in the vertical displacement is written analogously.

Using the table of inverse Laplace transforms in (3.3), we obtain

$$u_{xa} = \frac{a\,\mathrm{Im}}{2\pi\varepsilon R_0^2(p_1 - p_2)} \sum_{n=1}^{2} \int_0^{w_{0n}} \left\{ \frac{A_1(iw)}{\zeta - iv^{-1}} \sum_{k=1}^{2} \frac{1 - \exp\left[-p_k\left(t - \frac{wx - \eta z}{v}\right)\right]}{p_k} \times \right. \tag{3.4}$$
$$\times (-1)^{k-1} + \frac{aB_1(iw)}{\zeta - iv^{-1}} \sum_{k=1}^{2} \exp\left[-p_k\left(t - \frac{wx + \eta z}{v}\right)\right](-1)^{k-1} -$$
$$- \frac{A_2(iw)}{\zeta - iv^{-1}} \sum_{k=1}^{2} \frac{1 - \exp\left[-p_k\left(t - \frac{xw + \delta z}{v}\right)\right]}{p_k} (-1)^{k-1} +$$
$$\left. + \frac{aB_2}{\zeta - iv^{-1}} \sum_{k=1}^{2} \exp\left[-p_k\left(t - \frac{wx + \delta z}{w}\right)\right] \right\} dw + \frac{a\,\mathrm{Im}}{2\pi\varepsilon R_0^2(p_1 - p_2)} \times$$
$$\times \sum_{n=1}^{2} \int_{w_{0n}}^{\infty} \left\{ \frac{A_1(s)}{s - iv^{-1}} \sum_{k=1}^{2} \frac{1 - \exp\left[-p_k\left(t - \frac{zw(H + H^{-1})}{v}\right)\right]}{p_k} (-1)^{k-1} + \right.$$
$$+ \frac{aB_1(s)}{s - iv^{-1}} \sum_{k=1}^{2} \exp\left[-p_k\left(t - \frac{zw(H + H^{-1})}{v}\right)\right](-1)^{k-1} -$$
$$- \frac{A_2(s)}{s - iv^{-1}} \sum_{k=1}^{2} \frac{1 - \exp\left[p_k\left(t - \frac{wz(H + H)^{-1}}{v}\right)\right]}{p_k(-1)^{k-1}} -$$
$$\left. - a\frac{B_2}{s - iv^{-1}} \sum_{k=1}^{2} \exp\left[-p_k\left(t - \frac{wz(H + H^{-1})}{v}\right)\right](-1)^{k-1} \right\} ds$$

The expression (3.4) contains two kinds of members whose time variations differ qualitatively. Part of the members decrease with the course of time. The other part of the members tends to a constant as $t \to \infty$ and determines the static part of the displacement.

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